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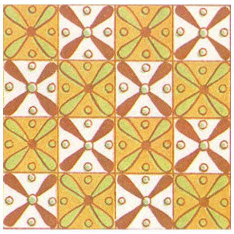
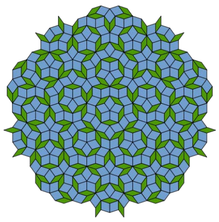
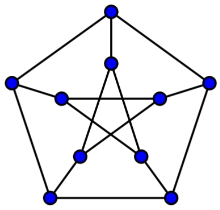
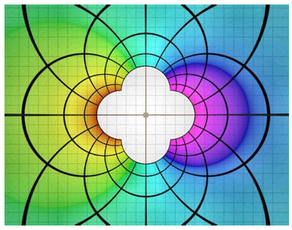
Wintersemester 2023/2024

HEGL Proseminar/Seminar “Illustrating Mathematics”: Groups and Symmetries

Geometry meets Algebra

Geometric structures are usually investigated by looking into their symmetries or lack thereof. In this iteration of the “Illustrating Mathematics” Proseminar/Seminar of the Heidelberg Experimental Geometry Lab (HEGL), we shall learn about some topics of geometry/topology, symmetries, and their connections with group theory. Students will deliver a short mathematical talk as well as work throughout the semester on visualization projects related to their chosen topics.

Organization and Format

The visualization projects will be carried out in small groups, supervised within HEGL. The project topics for the semester are distributed among the participants based on their suggestions and order of preference, *before* the Winter Semester starts. Throughout the semester, the groups will work on their visualization projects with the goal of producing a concrete, graphical output that will be part of HEGL’s library. This output can be, for example, a computer code / program, a (mobile or web) app, an illustrative animation, a 3D rendered model, a (physical) “handicraft” (*Bastel*, mit Bastelanleitung), a 3D printed object(s), etc. You can check out some examples of past graphical outputs at the lab’s website (https://hegl.mathi.uni-heidelberg.de/galleries/) or at the lab itself (in the `Mathematikon`, Level −1). The graphical output is presented at the end of the semester via both a short group presentation (5min) and a blog post at the HEGL homepage (https://hegl.mathi.uni-heidelberg.de/blog/).

Within each project topic, there are multiple themes for *individual* talks, so that each student will give an individual maths talk covering mathematical content related to their visualization projects. Preparing your lecture and actively attending all talks are *requirements* for successful participation in the seminar. All talks will take place on October 27 and 28.

Language: Presentations and material should be in English.

Target audience and choice of topics:

There are no formal prerequisites to enroll at the seminar besides familiarity with the usual first courses in Maths (such as “Analysis” and “lineare Algebra”). Some familiarity with Group Theory might be useful but is not a must. Similarly, some background with programming skills or maths software can be helpful, but these are also *not* required. Everyone with interest in symmetries, groups, and/or nice geometric objects is welcome to attend! If you would like to attend the seminar, please indicate your interest via Müsli (https://muesli.mathi.uni-heidelberg.de/) or email (`fschaffhauser@mathi.uni-heidelberg.de`).

Academic credits:

All academic credits earned through HEGL (Pro)seminars and Student Projects count as **Mathematics** credits **only**. In particular, HEGL activities are *not* part of the *Computer Science* Handbook, so they cannot count as academic credits towards this degree.

Project Topics

Project A: Groups = Symmetries

Algebraically, groups are nonempty sets with a given operation satisfying three axioms: associativity, existence of neutral element, and existence of inverses. However, every group can be realized as the set of symmetries of *some* object, sometimes very abstract (with not much structure to be seen) but sometimes very concretely, with lots of structure (e.g., symmetries of polygons, isometry groups of Riemannian manifolds, linear groups...). Groups themselves can also be realized as metric spaces. The aim of this project is to produce a visualization tool for finitely presented groups making use of their Cayley graphs.

(A.1) Groups as symmetries of objects
This talk discusses how groups can be seen as sets of symmetries of some objects, by means of examples (e.g., dihedral groups, free groups...). Some basic definitions and properties, such as groups itself and presentations should also be covered.
References: [2, Chapters 1 and 4], also some review from [2, Chapters 2, 5, and 27].

(A.2) Symmetric groups and Platonic solids
In this talk, we will see the most important example of finite groups: symmetric groups. These will then show up as the symmetry groups of the Platonic solids. We will also recall what homomorphisms and isomorphisms of groups are and why the symmetric groups are the most important example (Cayley’s theorem).
References: [2, Chapters 6 and 8], also some review from [2, Chapters 7 and 16]

(A.3) Group actions

The concept of a "group action" formalizes what it means for a group to be a set of symmetries of some object. In this talk we shall learn the language of actions, discussing basic definitions, orbits and stabilizers, examples, and the orbit-stabilizer theorem.
References: [2, Chapters 17 and 18]

(A.4) Cayley graphs and the Nielsen–Schreier theorem

There are different means to realize groups as geometric objects. One of the most prominent methods is via Cayley graphs. This talk introduces such graphs, discusses examples and elementary properties. We will also see the Nielsen–Schreier theorem.
References: [3, Chapter 3.2 and context], [2, Chapter 28]

Project B: Two-dimensional Euclidean Geometry

The Euclidean plane is one of our standard models for two-dimensional geometric phenomena. Its isometric transformations are important in multiple contexts and lead to beautiful patterns (nature, arts, pure mathematics...). The goal of this project is to showcase how isometries of the Euclidean planes behave.

(B.1) Matrix groups

Many objects can be embedded in (finite dimensional) vector spaces. It is thus of core interest to understand symmetries of such objects, which are encoded by matrix groups. This talk shall discuss the basics about matrix groups, examples (such as GL_n, SL_n, $\mathrm{O}_{p,q}$, isometries of $\mathbb{R}^n$), and potentially connections to Lie groups.
References: [2, Chapters 9 and 19].

(B.2) Isometries of the Euclidean plane

The plane $\mathbb{R}^2$ (with its usual Euclidean metric) is our standard model for planar geometry. This talk will discuss its distance-preserving transformations: definitions, examples (rotations, reflections, translations), and the algebraic structure of $\mathrm{Isom}(\mathbb{R}^2)$.
References: [2, Chapter 24] – see also the first half of [5, Chapter 19] for semidirect products.

(B.3) Wallpaper groups

"Wallpaper" patterns are ubiquitous in human culture, besides showing up in natural phenomena. This talk revolves around main concepts (e.g., lattices, fixed points, point-groups) and the complete classification of wallpaper groups.
References: [2, Chapters 25 and 26].

(B.4) Coxeter groups

This talk is an introduction to the theory of Coxeter groups, which are groups generated

by involutions. We will discuss the geometric realization of Coxeter groups as reflection groups and use this as a basis for the classification of finite Coxeter groups.
References: [4, Chapter I.1–I.4].

Project C: Two-dimensional Hyperbolic Geometry

The hyperbolic plane is less well-known, but it is still a standard model for two-dimensional geometric phenomena. The goal of this project is to showcase how isometries of hyperbolic planes behave, focusing on illustrating their similarities and differences from isometries of Euclidean planes.

(C.1) The Hyperbolic Plane
This talk introduces the hyperbolic plane $\mathbb{H}^2$ — standard models, how to move back and forth between them, description of geodesics, ideal points, and basic geometric features.
References: [1, Chapters 1 and 6.1], [6, Introduction].

(C.2) Möbius Transformations
Möbius transformations are maps from the complex numbers to themselves which preserve angles and map lines and circles to lines or circles. This talk introduces Möbius transformations in general, discusses their types and properties, fixed points, *etc.*
References: [1, Chapter 2], [6, Chapters 1 and 3.2].

(C.3) Lengths, distances and isometries
In the hyperbolic plane, one can measure distances. This talk will focus on lengths of paths in the hyperbolic plane and introduce the notion of a *geodesic path*. We will also also see that Möbius transformations are exactly the (orientation-preserving) isometries of the hyperbolic plane $\mathbb{H}^2$.
References: [1, Chapters 3 and 4].

(C.4) Convexity, area and trigonometry
In the hyperbolic plane, one can also measure areas and angles. This talk will focus on similarities and difference between the classical Euclidean plane and the hyperbolic plane, in particular the geometric properties of hyperbolic triangles and other polygons.
References: [1, Chapter 5] and [6, Chapter 2.2].

References

[1] *Anderson, J.*, Hyperbolic geometry. Springer, London (1999). https://link.springer.com/book/10.1007/978-1-4471-3987-4

[2] *Armstrong, M. A.*, Groups and symmetry. Springer, New York (1988). https://link.springer.com/book/10.1007/978-1-4757-4034-9

[3] *Löh, C.*, Geometric group theory – An introduction. Springer, Cham (2017). https://link.springer.com/book/10.1007/978-3-319-72254-2

[4] *Hiller, H.*, The Geometry of Coxeter groups, Pitman Pub. (1982). https://books.google.de/books/about/Geometry_of_Coxeter_Groups.html?id=7jzvAAAAMAAJ&redir_esc=y

[5] *Humphreys, J. F.*, A course in group theory. Oxford University Press (1996). https://www.amazon.de/-/en/John-F-Humphreys/dp/0198534590

[6] *Series, C.*, Hyperbolic geometry – Lecture notes at the University of Warwick (2008). https://homepages.warwick.ac.uk/~masbb/Papers/MA448.pdf