Approximate Metric 3D Embeddings of the 3-Sphere

HEGL project, WS 23/24

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Abstract

The topic of this project is to convert subsets of the 3-sphere into finite metric spaces in various ways and to explore structure-preserving 3D Euclidean embeddings for visualization. Besides differential geometry and parametrizations of the 3-sphere, this involves basic techniques from data analysis (multidimensional scaling, graph Laplacians and spectral embeddings; rigid registration of point clouds), numerical optimization and linear algebra, programming and visualization.

Organization. Either a single group up to 5 students works on this project, or few smaller groups solve subtasks and put together their results. Project participants choose their favorite software for numerical computation and rendering.

Visualizing the 3-sphere $S^3 \subset \mathbb{R}^4$ is not possible since our perception can only make sense of objects up to dimension 3. Therefore, one applies a mapping $\varphi\colon \mathbb{R}^4 \to \mathbb{R}^3$ in order to project an object of interest. A canonical choice is the *stereographic projection* which yields appealing graphical illustrations – see "Visualizing the Hopf fibration (HEGL project SS 2022)".

While stereographic projection preserves the topological structure and even is conformal, it suffers from distortion of the *metric* structure (Figure 1).

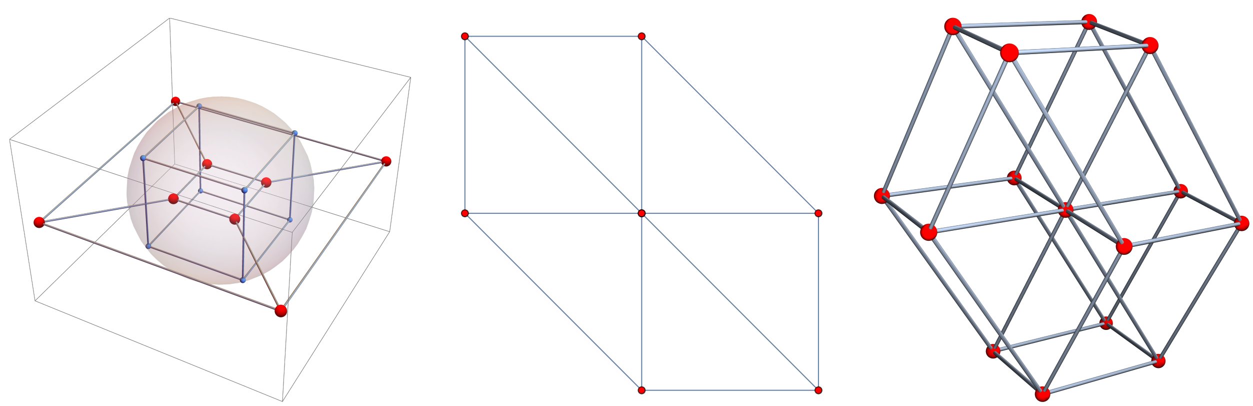

Figure 1: **Left:** Stereographic projection (red vertices) of the 3-cube (blue vertices) distorts the metric structure. **Center:** An alternative metric 2D embedding of the 3-cube which approximately preserves the metric structure. **Right:** An approximate metric 3D embedding of the 4-cube.

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